CIE3109
Structural Mechanics 4

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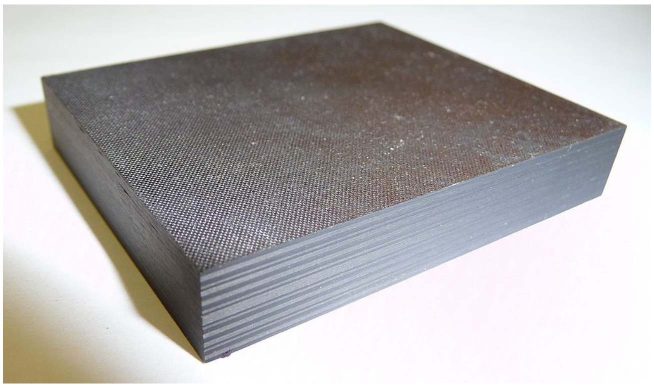
Module : Unsymmetrical and/or
inhomogeneous cross section

v2021

Question

*What is the **stress** (and strain) distribution in a cross section due to extension and bending (shear) in case of a unsymmetrical and/or non-homogeneous cross section' ?*

new smart materials in composite structures (MSc)

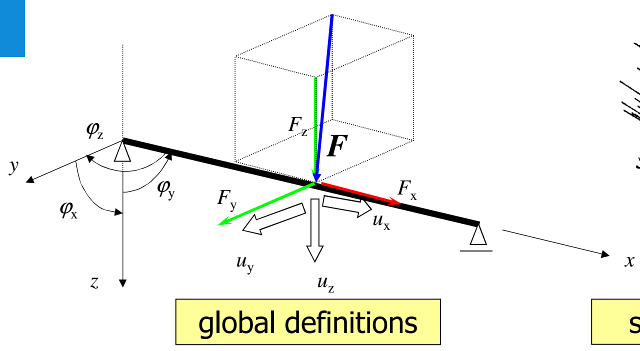


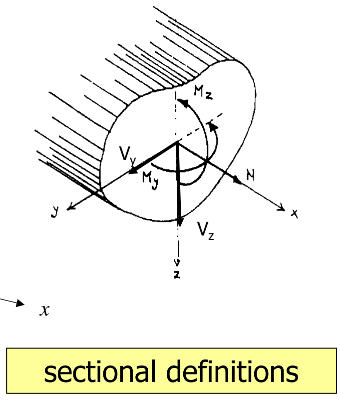
CIE3109 : Structural Mechanics 4

Lectures

- 1-2 Inhomogeneous and/or unsymmetrical cross sections
  - Introduction
  - General theory for extension and bending, beam theory
  - Unsymmetrical cross sections
    - Example curvature and loading
    - Example normalstress distribution
  - Deformations
- 3 Inhomogeneous cross sections
  - Refinement of the theory
  - Examples
- 4-5 Stresses and the core of the cross section
  - Normal stress in unsymmetrical cross section and the core
  - Shear stresses in unsymmetrical cross sections
  - Shear centre

BENDING and AXIAL LOADING

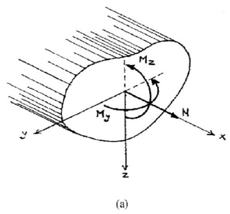




RELATION BETWEEN EXTERNAL LOADS AND DISPLACEMENTS (deformations)

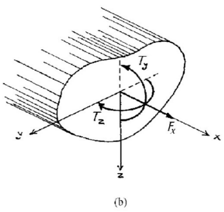
- Structural level
- Cross sectional level
- Micro level (material science)

Couples T and bending moments M

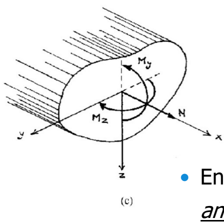


(a)

- Formal approach, positive bending moment M **in a cross section** acting *in plane* from positive side towards negative side.
- Formal approach **external** couple T, following right-hand-rule or corkscrew rule *around an axis*.



(b)

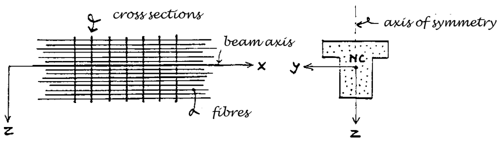


(c)

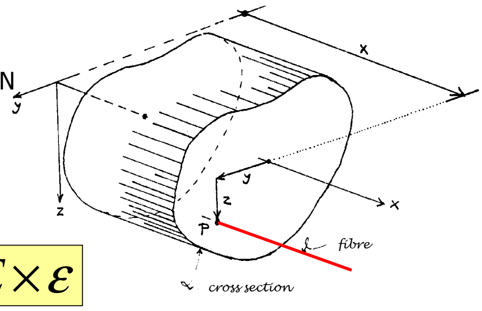
- Engineering practise with bending *around an axis* similar to definition of T on positive planes.

ASSUMPTIONS

- FIBRE MODEL



- SMALL ROTATIONS OF THE CROSS SECTION
- UNIAXIAL STRESS SITUATION IN THE FIBRES
- LINEAR ELASTIC MATERIAL BEHAVIOUR

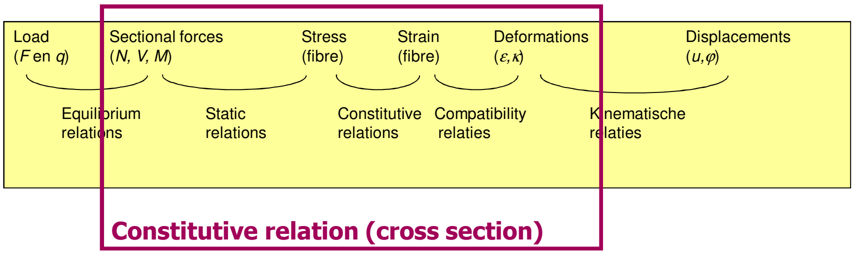


Hooke's law : $\sigma = E \times \varepsilon$

fibre in cross section x at position y,z

SOLUTION PATH

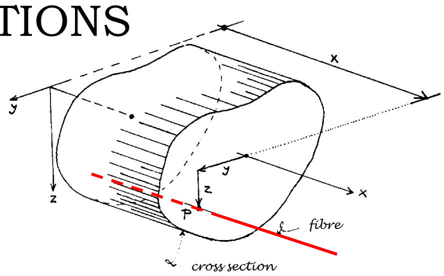
- Determine the strain distribution due to the displacements of the cross section?
- Determine the stress distribution caused by these strains?
- Determine the resulting forces in the cross section?

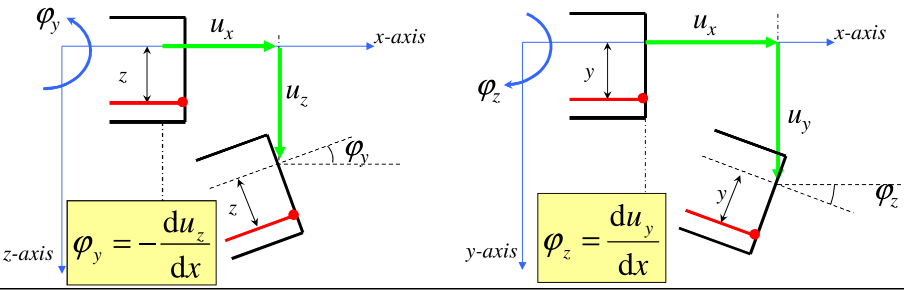


- Use the constitutive relation on structural level (slide 28)

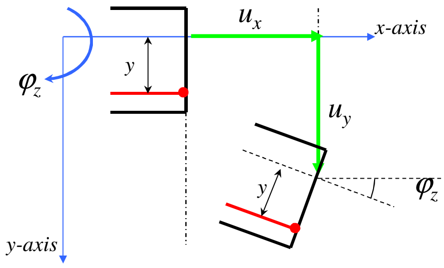
KINEMATIC RELATIONS

Relation between deformations and displacements





FIBRE MODEL



- Horizontal displacement

$$\varphi_y = -\frac{\mathrm{d}u_z}{\mathrm{d}x} \qquad \varphi_z = \frac{\mathrm{d}u_y}{\mathrm{d}x}$$

$$u(x,y,z) = u_x + z\times\varphi_y \;\; - y\times\varphi_z$$

or

$$u(x,y,z) = u_x - z\times u_z' \;\; - y\times u_y'$$

RELATIVE DISPLACEMENT-STRAIN

$$u(x,y,z) = u_x - y\times\varphi_z + z\times\varphi_y$$

or

$$u(x,y,z) = u_x - y\times u_y' - z\times u_z'$$

$$\varepsilon(y,z) = \frac{\partial u(x,y,z)}{\partial x}$$

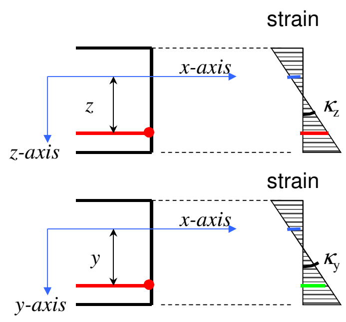
$$\varepsilon(y,z) = u_x' - y\times u_y'' - z\times u_z''$$

$$\varepsilon(y,z) = \varepsilon + y\times\kappa_y + z\times\kappa_z$$ with $\varepsilon = \frac{\mathrm{d}u_x}{\mathrm{d}x}$; $\kappa_y = -\frac{\mathrm{d}^2u_y}{\mathrm{d}x^2}$; $\kappa_z = -\frac{\mathrm{d}^2u_z}{\mathrm{d}x^2}$

STRAIN DISTRIBUTION

Conclusion:

Strain distribution is fully described with three deformation parameters

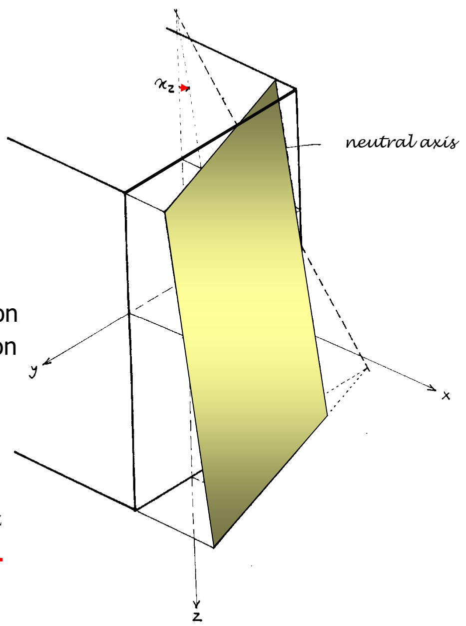


$$\varepsilon(y,z)=\varepsilon+y\times\kappa_y+z\times\kappa_z$$

STRAIN FIELD

Three parameters

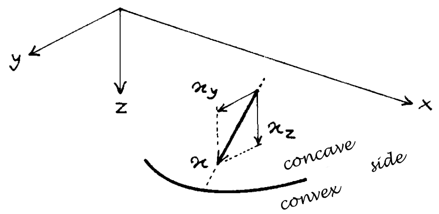
- strain in fibre which coincides with the *x*-axis
- slope of strain diagram in *y*-direction
- slope of strain diagram in *z*-direction



$$\varepsilon(y,z)=\varepsilon+y\times\kappa_y+z\times\kappa_z$$

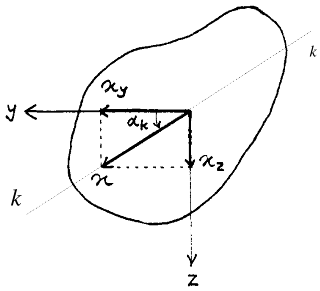
CURVATURE

- First order tensor
- Curvature in x-y-plane
- Curvature in x-z-plane
- Plane of curvature k

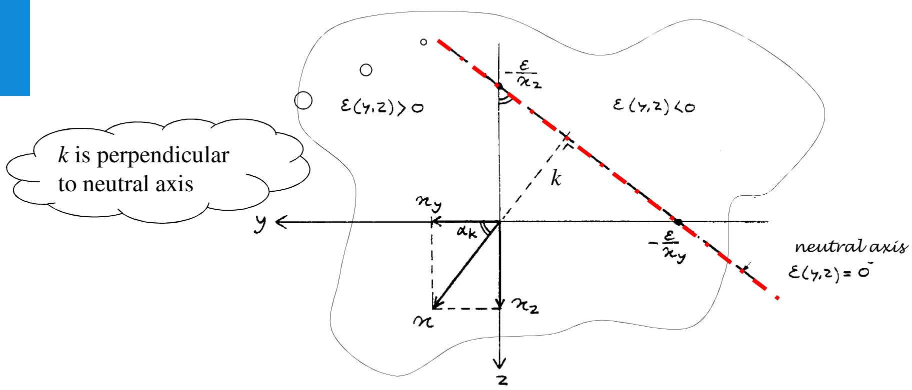


$$\kappa = \sqrt{\kappa_y^2 + \kappa_z^2}$$

$$\tan \alpha_k = \frac{\kappa_z}{\kappa_y}$$



NEUTRAL AXIS



$$\varepsilon(y,z) = \varepsilon + y \times \kappa_y + z \times \kappa_z = 0$$

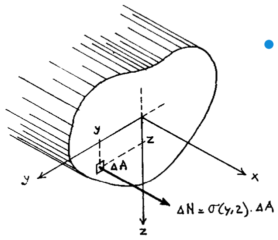
FROM STRAIN TO STRESS

- Constitutive relation : link between deformations and stresses

$$\sigma(y,z) = E(y,z) \times \varepsilon(y,z)$$

$$\sigma(y,z) = E(y,z) \times \left[\varepsilon + y \times \kappa_y + z \times \kappa_z \right]$$

STRESS AND SECTIONAL FORCES

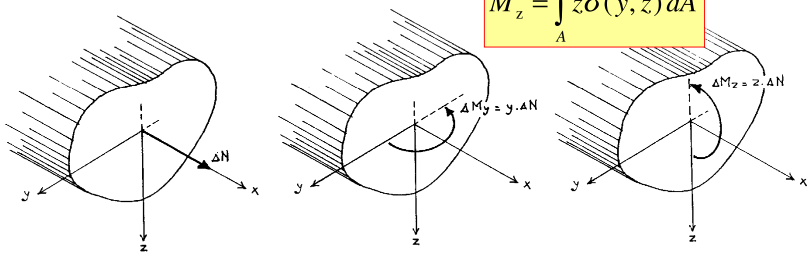


- Static relations : link between stresses and sectional forces

$$N = \int_A \sigma(y,z)\, dA$$

$$M_y = \int_A y\sigma(y,z)\, dA$$

$$M_z = \int_A z\sigma(y,z)\, dA$$



Elaborate ...

$$N = \int_A \sigma(y,z)dA = \int_A E(y,z)\times\left(\varepsilon + y\kappa_y + z\kappa_z\right)dA$$

$$M_y = \int_A y\sigma(y,z)dA = \int_A E(y,z)\times\left(\varepsilon + y\kappa_y + z\kappa_z\right)ydA$$

$$M_z = \int_A z\sigma(y,z)dA = \int_A E(y,z)\times\left(\varepsilon + y\kappa_y + z\kappa_z\right)zdA$$

$$N = \varepsilon\int_A E(y,z)dA + \kappa_y\int_A E(y,z)ydA + \kappa_z\int_A E(y,z)zdA \quad = EA\varepsilon \quad + ES_y\kappa_y \quad + ES_z\kappa_z$$

$$M_y = \varepsilon\int_A E(y,z)ydA + \kappa_y\int_A E(y,z)y^2dA + \kappa_z\int_A E(y,z)yzdA \quad = ES_y\varepsilon \quad + EI_{yy}\kappa_y \quad + EI_{yz}\kappa_z$$

$$M_z = \varepsilon\int_A E(y,z)zdA + \kappa_y\int_A E(y,z)yzdA + \kappa_z\int_A E(y,z)z^2dA \quad = ES_z\varepsilon \quad + EI_{yz}\kappa_y \quad + EI_{zz}\kappa_z$$

approach with "double-letter" symbols

CONSTITUTIVE RELATION

on cross sectional level

$$\begin{bmatrix} N \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EA & & \\ & EI_{yy} & EI_{yz} \\ & EI_{zy} & EI_{zz} \end{bmatrix} \begin{bmatrix} \varepsilon \\ \kappa_y \\ \kappa_z \end{bmatrix}$$

INDEPENDENT OF THE ORIGIN OF THE COORDINATE SYSTEM

SPECIAL LOCATION OF THE ORIGIN OF THE COORDINATE SYSTEM TO UNCOUPLE BENDING AND AXIAL LOADING

NORMAL FORCE CENTRE

- Bending and axial loading are uncoupled:

$$\begin{bmatrix} N \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EA & 0 & 0 \\ 0 & EI_{yy} & EI_{yz} \\ 0 & EI_{zy} & EI_{zz} \end{bmatrix} \begin{bmatrix} \varepsilon \\ \kappa_y \\ \kappa_z \end{bmatrix}$$

ε → Axial loading

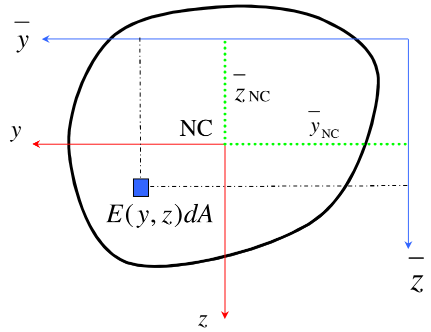
κ_y, κ_z → Bending

definition NC : $ES_y = ES_z = 0$

if N = 0 then zero strain ε at the NC and the $n.a.$ runs through the NC

LOCATION NC

$\bar{y} = y + \bar{y}_{NC} \qquad \bar{z} = z + \bar{z}_{NC}$



$$ES_{\bar{y}} = \int_A E(y,z) \times \bar{y} dA =$$

$$\int_A E(y,z) \times y dA + \bar{y}_{NC} \int_A E(y,z) dA = \cancel{ES_y} + EA \times \bar{y}_{NC}$$

$$ES_{\bar{z}} = \int_A E(y,z) \times \bar{z} dA =$$

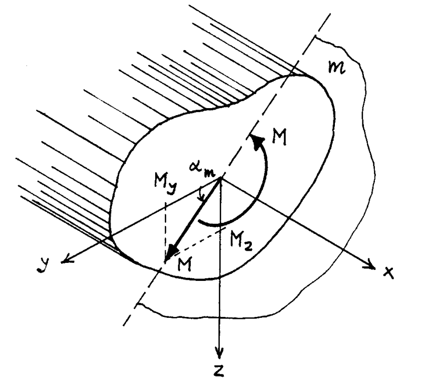
$$\int_A E(y,z) \times z dA + \bar{z}_{NC} \int_A E(y,z) dA = \cancel{ES_z} + EA \times \bar{z}_{NC}$$

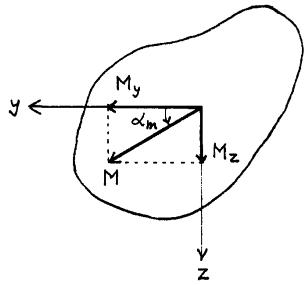
$$\bar{y}_{NC} = \frac{ES_{\bar{y}}}{EA}$$

$$\bar{z}_{NC} = \frac{ES_{\bar{z}}}{EA}$$

RESULT

- Bending and axial loading are uncoupled
- Moment is first order tensor

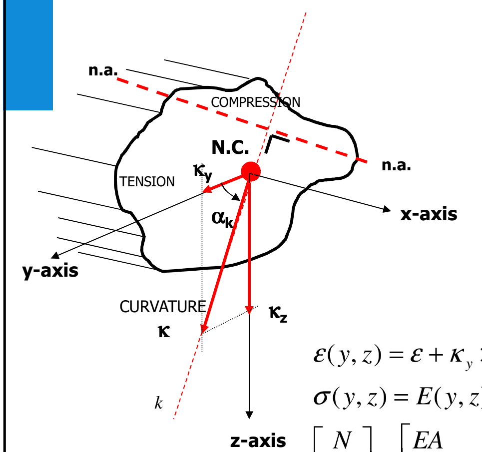


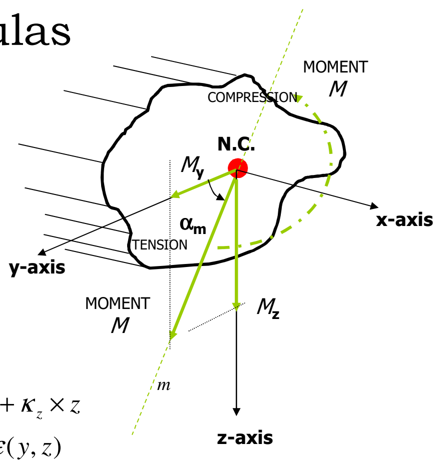


$$M = \sqrt{M_y^2 + M_z^2}$$

$$\tan \alpha_m = \frac{M_z}{M_y}$$

SUMMARY of formulas





$$\varepsilon(y,z) = \varepsilon + \kappa_y \times y + \kappa_z \times z$$

$$\sigma(y,z) = E(y,z) \times \varepsilon(y,z)$$

$$\begin{bmatrix} N \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EA & & \\ & EI_{yy} & EI_{yz} \\ & EI_{yz} & EI_{zz} \end{bmatrix} \times \begin{bmatrix} \varepsilon \\ \kappa_y \\ \kappa_z \end{bmatrix}$$

PLANE OF LOADING SAME AS PLANE OF CURVATURE ?

$$\overline{M} = \lambda \overline{\kappa}$$

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \lambda \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} \Leftrightarrow$$

$$\begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix} \times \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} = \lambda \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} \Leftrightarrow$$

$$\begin{bmatrix} EI_{yy} - \lambda & EI_{yz} \\ EI_{zy} & EI_{zz} - \lambda \end{bmatrix} = 0 \quad \text{eigenvalue problem}$$

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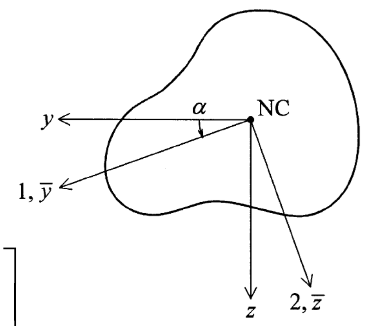
PRINCIPAL DIRECTIONS



$$\begin{bmatrix} M_{\bar{y}} \\ M_{\bar{z}} \end{bmatrix} = \begin{bmatrix} EI_{\bar{y}\bar{y}} & 0 \\ 0 & EI_{\bar{z}\bar{z}} \end{bmatrix} \begin{bmatrix} \kappa_{\bar{y}} \\ \kappa_{\bar{z}} \end{bmatrix}$$

$$EI_{\bar{y}\bar{y},\bar{z}\bar{z}} = \tfrac{1}{2}\left(EI_{yy} + EI_{zz}\right) \pm \sqrt{\left(\tfrac{1}{2}(EI_{yy} - EI_{zz})\right)^2 + EI_{yz}^2}$$

$$\tan 2\alpha_{\bar{y},\bar{z}} = \frac{EI_{yz}}{\tfrac{1}{2}(EI_{yy} - EI_{zz})}$$

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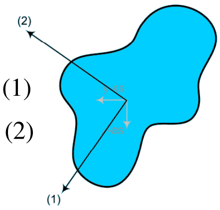
ROTATE TO PRINCIPAL DIRECTIONS

$\bar{z} =$ direction (1)

$\bar{y} =$ direction (2)



$$\begin{bmatrix} M_{\bar{y}} \\ M_{\bar{z}} \end{bmatrix} = \begin{bmatrix} EI_{\bar{y}\bar{y}} & 0 \\ 0 & EI_{\bar{z}\bar{z}} \end{bmatrix} \begin{bmatrix} \kappa_{\bar{y}} \\ \kappa_{\bar{z}} \end{bmatrix} \Leftrightarrow \begin{matrix} M_{\bar{y}} = EI_{\bar{y}\bar{y}} \kappa_{\bar{y}} \\ M_{\bar{z}} = EI_{\bar{z}\bar{z}} \kappa_{\bar{z}} \end{matrix}$$

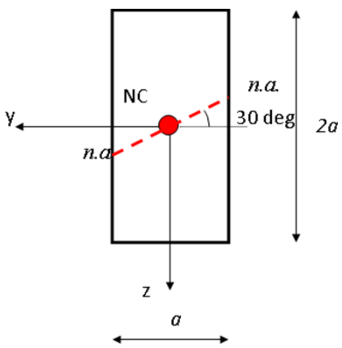
$$\tan(\bar{\alpha}_{\text{m}}) = \frac{M_{\bar{z}}}{M_{\bar{y}}} = \frac{EI_{\bar{z}\bar{z}} \kappa_{\bar{z}}}{EI_{\bar{y}\bar{y}} \kappa_{\bar{y}}} = \frac{EI_{\bar{z}\bar{z}}}{EI_{\bar{y}\bar{y}}} \tan(\bar{\alpha}_{\text{k}}) \qquad \bar{\alpha}_{\text{m}} = \bar{\alpha}_{\text{k}} \;??$$

1) $\bar{\alpha}_{\text{m}} = \bar{\alpha}_{\text{k}} = 0 \quad =$ principal direction

2) $\bar{\alpha}_{\text{m}} = \bar{\alpha}_{\text{k}} = \pi / 2 =$ other principal direction

3) $EI_{\bar{y}\bar{y}} = EI_{\bar{z}\bar{z}}$

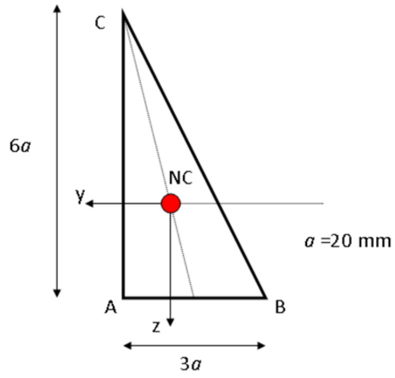
EXAMPLE 1



Determine the position of the plane loading *m-m* for the specified position of the neutral line which makes an angle of 30º degrees with the *y*-axis. The rectangular cross section is homogeneous.

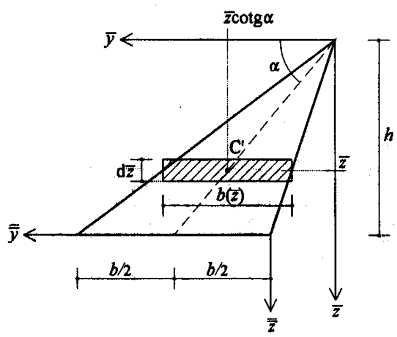
EXAMPLE 2

A triangular cross section is loaded by pure bending only. The cross section is homogeneous and the Youngs modulus is *E*. The normal stress in point A and C is equal to 10 N/mm².



a) Calculate the magnitude and direction of the resulting bending moment
b) Compute the stress in point B

F1 = HELP

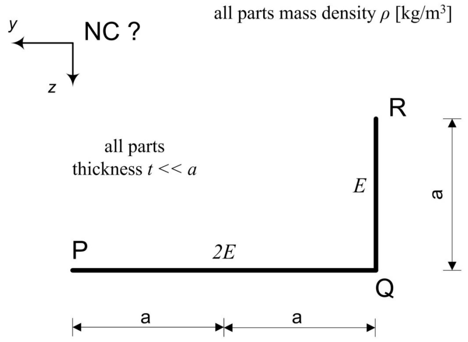


$$I_{zz} = \tfrac{1}{36} bh^3$$

$$I_{yy} = \tfrac{1}{48} b^3 h + \frac{\tfrac{1}{36} bh^3}{(\tan\alpha)^2}$$

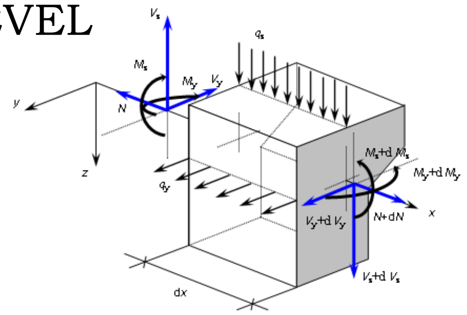
$$I_{yz} = \tfrac{1}{36} \frac{bh^3}{\tan\alpha}$$

EXAMPLE 3



a) Find the location of the NC
b) Compute all cross sectional properties
c) Find the location of the centre of gravity

STRUCTURAL LEVEL
Equillibrium conditions



$$\frac{\mathrm{d}N}{\mathrm{d}x}+q_x=0 \qquad\qquad \Rightarrow \quad \frac{\mathrm{d}N}{\mathrm{d}x}=-q_x$$

$$\frac{\mathrm{d}V_y}{\mathrm{d}x}+q_y=0 \quad \text{and} \quad \frac{\mathrm{d}M_y}{\mathrm{d}x}-V_y=0 \quad \Rightarrow \quad \frac{\mathrm{d}^2M_y}{\mathrm{d}x^2}=-q_y$$

$$\frac{\mathrm{d}V_z}{\mathrm{d}x}+q_z=0 \quad \text{and} \quad \frac{\mathrm{d}M_z}{\mathrm{d}x}-V_z=0 \quad \Rightarrow \quad \frac{\mathrm{d}^2M_z}{\mathrm{d}x^2}=-q_z$$

STRUCTURAL LEVEL
Differential Equations

Kinematics:

$$\varepsilon = \frac{\mathrm{d}u_x}{\mathrm{d}x} = u_x' \qquad \kappa_y = -\frac{\mathrm{d}^2 u_y}{\mathrm{d}x^2} = -u_y'' \qquad \kappa_z = -\frac{\mathrm{d}^2 u_z}{\mathrm{d}x^2} = -u_z''$$

$$-EAu_x'' = q_x \quad \text{axial loading}$$

$$EI_{yy}u_y'''' + EI_{yz}u_z'''' = q_y$$
$$EI_{yz}u_y'''' + EI_{zz}u_z'''' = q_z \quad \text{bending}$$

$$\frac{\mathrm{d}N}{\mathrm{d}x} = -q_x \qquad \frac{\mathrm{d}^2 M_y}{\mathrm{d}x^2} = -q_y \qquad \frac{\mathrm{d}^2 M_z}{\mathrm{d}x^2} = -q_z$$

Result

$$u_x'' = -\frac{q_x}{EA}$$

$$Q_y = \frac{EI_{zz}EI_{yy}q_y - EI_{yz}EI_{yy}q_z}{EI_{yy}EI_{zz} - EI_{yz}^2}$$

$$Q_z = \frac{-EI_{yz}EI_{zz}q_y + EI_{yy}EI_{zz}q_z}{EI_{yy}EI_{zz} - EI_{yz}^2}$$

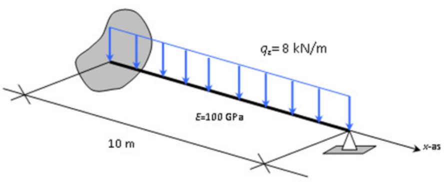
$$EAu_x'' = -q_x$$

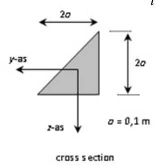
$$EI_{yy}u_y'''' = Q_y$$
$$EI_{zz}u_z'''' = Q_z$$

modify the load in all *forget-me-not's*
***OR** solve differential equation*

EXAMPLE (see lecture notes)

$$EI_{i,j} = \tfrac{1}{9} Ea^4 \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$





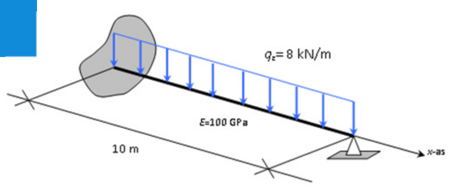
$$x = 0: \quad (u_y = 0;\, u_z = 0;\, \varphi_y = 0;\; \varphi_z = 0)$$

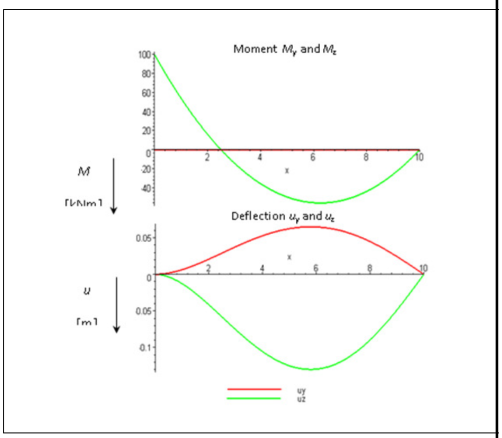
$$x = l: \quad (u_y = 0;\, u_z = 0;\, M_y = 0;\, M_z = 0)$$

$$u_y = C_1 + C_2 x + C_3 x^2 + C_4 x^3 - \frac{q_z x^4}{16 E a^4}$$

$$u_z = D_1 + D_2 x + D_3 x^2 + D_4 x^3 + \frac{q_z x^4}{8 E a^4}$$

Result





SUMMARY displacements

Original y-z coordinate system

- solve differential equations
- Use "pseudo" load for y- and z-direction and use this load in standard engineering equations
- Use curvature distribution in combination with moment area theoremes to obtain displacements and rotations (see notes)

Principal 1-2 coordinate system

- Use principal directions, decompose load in (1) and (2) direction, and compute displacements in (1) and (2) direction with standard engineering equations. Finally transformate the displacements back to y- and z-direction

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